

Unitary relations in time-dependent harmonic oscillators

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1999 J. Phys. A: Math. Gen. 32 3449

(<http://iopscience.iop.org/0305-4470/32/18/318>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.105

The article was downloaded on 02/06/2010 at 07:30

Please note that [terms and conditions apply](#).

Unitary relations in time-dependent harmonic oscillators

Dae-Yup Song[†]

Department of Physics, Suncheon National University, Suncheon 540-742, Korea

Received 20 October 1998

Abstract. For a harmonic oscillator with time-dependent (positive) mass and frequency, a unitary operator is shown to transform the quantum states of the system to those of a harmonic oscillator system of unit mass and time-dependent frequency, as well as operators. For a driven harmonic oscillator, a unitary transformation which relates the driven system and a system of the same mass and frequency without a driving force is given, as a generalization of previous results, in terms of the solution of classical equation of motion of the driven system. Thus transformations give a simple way of finding exact wavefunctions of a driven harmonic oscillator system, provided the quantum states of the corresponding system of unit mass are given.

1. Introduction

The harmonic oscillators with time-dependent mass and frequency have long been of interest and give examples of exactly solvable time-dependent systems. For the oscillator of constant mass and time-dependent frequency, Lewis [1, 2] has shown that there exists a quantum mechanically invariant operator, unaware of Ermakov's results [3]. This so-called Ermakov–Lewis invariant operator can be used to find exact quantum states. This method has then been generalized to include time-dependent mass [4, 5], driving force [6], and to a general quadratic system whose Hamiltonian has all terms of position and momentum quadratic or less than that [7, 8].

Another systematic method to find exact quantum states of the systems is to use the Lagrangian formulation of Feynman and Hibbs [9] who have shown that the position-dependent part of the kernel (propagator) is determined from the classical action. This observation by Feynman and Hibbs gives a good explanation of the fact that the wavefunctions of the quantum states are described in terms of solutions of classical equation of motion. In [10], this method has been developed to give the exact kernel. The wavefunctions of general quadratic systems are then found by factorizing the kernel.

With these generalizations from Lewis's results, one important question arises: do the generalizations give quite new systems? This question has long been studied through the canonical transformation in classical mechanics [11, 12]. In a quantum treatment [13, 14], in addition to the recognition of the relation between a driven and an undriven system [15, 16], a part of the answer to this question has been given by Mostafazadeh [14]. He found a unitary operator which transforms the Hamiltonian of the oscillator of time-dependent mass and frequency to that of constant mass. So, one of his conclusions was confirmation of the old (classical) result, in quantum treatment, that the Hamiltonian of the Caldirola–Kanai (CK) system [17, 18] can be obtained from that of a simple harmonic oscillator [19].

[†] E-mail address: dsong@suncheon.suncheon.ac.kr

The purpose of this paper is to show that the generalizations [4–8, 10] of Lewis's results can be done through unitary transformations not only at the operator level but also in representation theory. For this, we need two unitary transformations. One of the transformations is to relate the driven harmonic oscillator system to that of the same parameters without a driving force. The operator of this transformation will be given in terms of the solution of the classical equation of motion of the driven system, as a generalization of previous results [15, 16, 19]. The other transformation is to change the mass and frequency of the system. The mass–frequency relation given by Mostafazadeh [14] will also be obtained by comparing the classical equations of motion of the two systems. If we choose proper parameters which will be found explicitly, the transformation changes the system of time-dependent (positive) mass and frequency to that of unit mass.

By applying the operators to the quantum states of the system of unit mass, it will be shown that the wavefunctions of the driven harmonic oscillator can be obtained from those of the corresponding undriven system of unit mass. Therefore, this transformation method gives a simple way of finding exact quantum states of a driven harmonic oscillator system [6] or a general quadratic system [7, 8, 10], provided quantum states of the corresponding system of unit mass are given. As explicit examples, we consider two models which are equivalent to simple harmonic oscillators. One of them is the CK system [17, 18] and the wavefunctions of this system will be evaluated from those of simple harmonic oscillators.

2. The unitary transformations for harmonic oscillator systems without a driving force

We start with the transformation for the time-dependent Hamiltonian

$$H(p, x, t) = \frac{p^2}{2M(t)} + \frac{1}{2}M(t)w^2(t)x^2 \quad (1)$$

where $M(t)$ and $w(t)$ are the time-dependent (positive) mass and frequency, respectively. Then the wavefunction $\psi(x, t)$ of a quantum eigenstate should satisfy the Schrödinger equation

$$O\psi(x, t) = 0 \quad \text{with} \quad O \equiv -i\hbar \frac{\partial}{\partial t} + H\left(\frac{\hbar}{i} \frac{\partial}{\partial x}, x, t\right). \quad (2)$$

Since we will consider the time-dependent unitary transformation, it is necessary to consider the transformation of the operator O instead of H [13, 14, 20]. With the unitary operator, U_c , defined as

$$U_c = e^{i\alpha x^2/\hbar} e^{i\beta(xp+px)/4\hbar} \quad (3)$$

one may find the relation

$$U_c O U_c^\dagger = -i\hbar \frac{\partial}{\partial t} + \frac{p^2}{2Me^\beta} + (xp + px) \left[-\frac{\dot{\beta}}{4} - \frac{\alpha}{Me^\beta} \right] + \frac{x^2}{2} \left[Mw^2e^\beta + 2\alpha\dot{\beta} - 2\dot{\alpha} + \frac{4\alpha^2}{Me^\beta} \right] \quad (4)$$

where the dots over variables denote the differentiation with respect to time. Equation (4) implies that the unitary transformation gives rise to a new system described by the Hamiltonian

$$H_{new} = \frac{p^2}{2Me^\beta} + (xp + px) \left[-\frac{\dot{\beta}}{4} - \frac{\alpha}{Me^\beta} \right] + \frac{x^2}{2} \left[Mw^2e^\beta + 2\alpha\dot{\beta} - 2\dot{\alpha} + \frac{4\alpha^2}{Me^\beta} \right]. \quad (5)$$

As is well known, the term proportional to $(xp + px)$ in Hamiltonian can be generated by acting a unitary transformation in the Hamiltonian formulation [21], or by adding a term proportional

to dx^2/dt to the Lagrangian [10]. Since the term proportional to $(xp + px)$ can be interpreted as a result of simple unitary transformation, we will take α as

$$\alpha = -\frac{1}{4}M\dot{\beta}e^{\beta}. \tag{6}$$

With this relation, H_{new} is written as

$$H_{new} = \frac{p^2}{2Me^{\beta}} + Me^{\beta} \left[w^2 + \frac{1}{2} \frac{\dot{M}}{M} \dot{\beta} + \frac{\ddot{\beta}}{2} + \frac{\dot{\beta}^2}{4} \right] \frac{x^2}{2}. \tag{7}$$

H_{new} in equation (7) shows [14, 20] that a unitary transformation can be used to find a new harmonic oscillator system which has a different mass and frequency from the original system of equation (1). Among these systems, we can find a system of unit mass by taking

$$\beta = -\ln M(t) \tag{8}$$

which is described by the Hamiltonian

$$H_0 = \frac{p^2}{2} + \frac{1}{2} \left(w^2 + \frac{1}{4} \left(\frac{\dot{M}}{M} \right)^2 - \frac{1}{2} \frac{\ddot{M}}{M} \right) x^2 = \frac{p^2}{2} + \frac{1}{2} \left(w^2 - \frac{1}{\sqrt{M}} \frac{d^2\sqrt{M}}{dt^2} \right) x^2. \tag{9}$$

That is, the mass of the system is 1, while the new frequency, w_0 , is given by [14]

$$w_0^2(t) = w^2 - \frac{1}{\sqrt{M}} \frac{d^2\sqrt{M}}{dt^2}. \tag{10}$$

The unitary operator for the transformation from the Hamiltonian in equation (1) to H_0 is now given as

$$U_0 = \exp\left(\frac{i}{4\hbar} \frac{\dot{M}}{M} x^2\right) \exp\left(-\frac{i \ln M}{4\hbar} (xp + px)\right). \tag{11}$$

In the above equations, the unit mass which has not been written explicitly should be taken into account to find the correct physical dimensions, which will also be true from now on.

One may find that the unitary operator in equation (11) [14] which does not depend on the solutions of the classical equation of motion is different from that in [13].

The system described by Hamiltonian in equation (9) is one of those considered by Lewis [1]. With non-negative integer n , the n -order Hermite polynomial H_n and two linearly independent real solutions $u_0(t), v_0(t)$ of the classical equation of motion

$$\ddot{x}_0 + w_0^2(t) \bar{x}_0 = 0 \tag{12}$$

the wavefunctions of the quantum eigenstates are given as [1, 4, 5, 10]

$$\begin{aligned} \psi_n^0(x, t) &= \frac{1}{\sqrt{2^n n!}} \left(\frac{\Omega_0}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{\rho_0(t)}} \left[\frac{u_0(t) - i v_0(t)}{\rho_0(t)} \right]^{n+1/2} \exp \left[\frac{x^2}{2\hbar} \left(-\frac{\Omega_0}{\rho_0^2(t)} + i \frac{\dot{\rho}_0(t)}{\rho_0(t)} \right) \right] \\ &\times H_n \left(\sqrt{\frac{\Omega_0}{\hbar}} \frac{x}{\rho_0(t)} \right). \end{aligned} \tag{13}$$

In equation (13), Ω_0 and $\rho_0(t)$ are defined as

$$\Omega_0 = [\dot{v}_0(t) u_0(t) - \dot{u}_0(t) v_0(t)] \quad \rho_0(t) = \sqrt{u_0^2(t) + v_0^2(t)}. \tag{14}$$

Ω_0 which depends on the choice of classical solutions is constant along the time evolution. Even though the corresponding Schrödinger equation is formally satisfied for any non-zero Ω_0 , we will only consider the cases of positive Ω_0 for applications.

For the simple harmonic oscillator of unit mass and positive constant frequency w_s , one may take the classical solutions as $u_0 = A \cos w_s t$, and $v = B \sin w_s t$, with positive constants A and B . The wavefunctions in equation (13) then becomes

$$\begin{aligned} \psi_n^{SHO}(w_s; x, t) &= \frac{1}{\sqrt{2^n n!}} \left(\frac{C w_s}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{\tilde{\rho}_s(t)}} \left[\frac{C \cos w_s t - i \sin w_s t}{\tilde{\rho}_s(t)} \right]^{n+1/2} \\ &\times \exp \left(\frac{x^2}{2\hbar} \left[-\frac{C w_s}{\tilde{\rho}_s^2(t)} + i \frac{\dot{\tilde{\rho}}_s(t)}{\tilde{\rho}_s(t)} \right] \right) H_n \left(\sqrt{\frac{C w_s}{\hbar}} \frac{x}{\tilde{\rho}_s(t)} \right) \end{aligned} \quad (15)$$

where

$$\tilde{\rho}_s(w_0) = \sqrt{1 + (C^2 - 1) \cos^2 w_0 t} \quad \text{and} \quad C = \frac{A}{B}. \quad (16)$$

With the choice of $C = 1$, $\psi_n^{SHO}(w_s, x, t)$ reduces to the usual stationary wavefunction of the unit mass simple harmonic oscillator; however, for $C \neq 1$, the wavefunctions describe the quantum eigenstates of pulsating probability distribution.

The unitary transformation changes quantum states as well as operators. To show this fact explicitly, we define a set of two linearly independent functions $\{u, v\}$ as

$$u(t) = \frac{u_0(t)}{\sqrt{M}} \quad v(t) = \frac{v_0(t)}{\sqrt{M}}. \quad (17)$$

One then easily finds that $\{u, v\}$ satisfies the differential equation

$$\frac{d}{dt}(M\dot{\bar{x}}) + M(t)w^2(t)\bar{x} = 0 \quad \text{or} \quad \ddot{\bar{x}} + \frac{\dot{M}}{M}\dot{\bar{x}} + w^2(t)\bar{x} = 0 \quad (18)$$

which is the classical equation of motion for the system described by the Hamiltonian in equation (1). Furthermore, by substituting \bar{x} with \bar{x}_0/\sqrt{M} in equation (18) and comparing equations (12) and (18), one may reproduce the mass–frequency relation (10). We also define $\Omega, \rho(t)$ as

$$\Omega = M(t)[\dot{v}(t)u(t) - \dot{u}(t)v(t)] \quad \rho(t) = \sqrt{u^2(t) + v^2(t)}. \quad (19)$$

Ω is then constant along time. Making use of the fact that

$$e^{(a(t)x(\partial/\partial x))} f(x) = f(e^{a(t)}x) \quad (20)$$

through the unitary transformation, one may find the wavefunction for the system of the Hamiltonian in equation (1):

$$\begin{aligned} \psi_n(x, t) &= U_0^\dagger \psi_n^0 \\ &= \frac{1}{\sqrt{2^n n!}} \left(\frac{\Omega}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{\rho(t)}} \left[\frac{u(t) - iv(t)}{\rho(t)} \right]^{n+1/2} \exp \left[\frac{x^2}{2\hbar} \left(-\frac{\Omega}{\rho^2(t)} + iM(t)\frac{\dot{\rho}(t)}{\rho(t)} \right) \right] \\ &\times H_n \left(\sqrt{\frac{\Omega}{\hbar}} \frac{x}{\rho(t)} \right) \end{aligned} \quad (21)$$

$$\quad (22)$$

which agrees with the known result [4, 5, 10].

3. Examples

We consider two systems which are unitarily equivalent to the simple harmonic oscillator, as examples. The first one is the CK system [17, 18] described by the Hamiltonian

$$H^{CK}(p, x, t) = \frac{p^2}{2me^{\gamma t}} + \frac{1}{2}me^{\gamma t}w_1^2x^2 \quad (23)$$

with constant m, γ and w_1 . Equation (10) shows that the CK system is unitarily equivalent to the simple harmonic oscillator of unit mass and constant frequency w_{ck} , where w_{ck} is given by [19]

$$w_{ck}^2 = w_1^2 - \frac{1}{4}\gamma^2. \tag{24}$$

For the case of positive real w_{ck} , the wavefunctions are easily found from those in equation (15) by applying the relation in (21),

$$\begin{aligned} \psi_n^{CK} &= \exp\left(\frac{i}{4\hbar}(\gamma t + \ln m)(xp + px)\right) \exp\left(-\frac{i\gamma}{4\hbar}x^2\right) \psi_n^{SHO}(w_{ck}; x, t) \\ &= (me^{\gamma t})^{1/4} \exp\left(\frac{\gamma t + \ln m}{2}x \frac{\partial}{\partial x}\right) \exp\left(-\frac{i\gamma}{4\hbar}x^2\right) \psi_n^{SHO}(w_{ck}; x, t) \\ &= \frac{1}{\sqrt{2^n n!}} \left(\frac{me^{\gamma t} C w_{ck}}{\pi \hbar}\right)^{1/4} \frac{1}{\sqrt{\tilde{\rho}_{ck}}} \left[\frac{C \cos w_{ck} t - i \sin w_{ck} t}{\tilde{\rho}_{ck}}\right]^{n+1/2} \\ &\quad \times \exp\left[\frac{me^{\gamma t} x^2}{2\hbar} \left(-\frac{C w_{ck}}{\tilde{\rho}_{ck}^2} + i\left(\frac{\dot{\tilde{\rho}}_{ck}}{\tilde{\rho}_{ck}} - \frac{\gamma}{2}\right)\right)\right] H_n\left(\sqrt{\frac{me^{\gamma t} C w_{ck}}{\hbar}} \frac{x}{\tilde{\rho}_{ck}}\right) \end{aligned} \tag{25}$$

where

$$\tilde{\rho}_{ck} = \tilde{\rho}_s(w_{ck}). \tag{26}$$

By adjusting C , the wavefunctions in equation (25) can be shown to give those in [22–25]. By taking two linearly independent solution of the classical equation of motion:

$$\ddot{x} + \gamma \dot{x} + w^2(t)x = 0$$

of the CK system as $u = Ae^{-\gamma t/2} \cos w_{ck} t$ and $v = Be^{-\gamma t/2} \sin w_{ck} t$, one can also obtain the wavefunctions in equation (25) from equation (22).

As another example, we consider the system of the damped pulsating oscillator considered in [6, 26], where the time-dependent mass M_{Lo} is given as $M_{Lo} = m_0 \exp[2(\gamma t + \mu \sin \nu t)]$ with constant m_0, γ, μ and ν . The frequency $w(t)$ of the model is defined as $w^2 = w_{Lo}^2 + (1/\sqrt{M_{Lo}})(d^2 \sqrt{M_{Lo}}/dt^2)$, with constant w_{Lo} . Though this model looks complicated, equation (10) implies that this system is unitarily equivalent to the simple harmonic oscillator of unit mass and constant frequency w_{Lo} . The wavefunctions can also be obtained from those in equation (15) as

$$\begin{aligned} \psi_n^{Lo} &= \frac{1}{\sqrt{2^n n!}} \left(\frac{M_{Lo} C w_{Lo}}{\pi \hbar}\right)^{1/4} \frac{1}{\sqrt{\tilde{\rho}_{Lo}}} \left[\frac{C \cos w_{Lo} t - i \sin w_{Lo} t}{\tilde{\rho}_{Lo}}\right]^{n+1/2} \\ &\quad \times \exp\left[\frac{M_{Lo}}{2\hbar} x^2 \left(-\frac{C w_{Lo}}{\tilde{\rho}_{Lo}^2} + i\left(\frac{\dot{\tilde{\rho}}_{Lo}}{\tilde{\rho}_{Lo}} - \frac{1}{2} \frac{\dot{M}_{Lo}}{M_{Lo}}\right)\right)\right] H_n\left(\sqrt{\frac{M_{Lo} C w_{Lo}}{\hbar}} \frac{x}{\tilde{\rho}_{Lo}}\right) \end{aligned} \tag{27}$$

where

$$\tilde{\rho}_{ck} = \tilde{\rho}_s(w_{Lo}). \tag{28}$$

4. The transformations for driven oscillator systems

The driven harmonic oscillator is described by the Hamiltonian

$$H^F = \frac{P^2}{2M(t)} + \frac{1}{2}M(t) w^2(t) x^2 - x F(t). \tag{29}$$

To find the unitary transformation, we define the x_p as a particular solution of the classical equation of motion:

$$\frac{d}{dt}(M\dot{x}_p) + M(t)w^2(t)x_p = F(t). \quad (30)$$

We also introduce a function $\delta(t)$ defined as

$$\delta = \frac{1}{2}Mw^2x_p^2 - \frac{1}{2}M\dot{x}_p^2. \quad (31)$$

By defining an operator O_F as

$$O_F = -i\hbar \frac{\partial}{\partial t} + H_F \quad (32)$$

making use of the equations (30) and (31), one can find the relation

$$U_F O U_F^\dagger = O_F \quad (33)$$

where U_F is given as

$$U_F = \exp\left[\frac{i}{\hbar}(M\dot{x}_p x + \delta(t))\right] \exp\left(-\frac{i}{\hbar}x_p p\right). \quad (34)$$

The wavefunction for the system of the Hamiltonian in equation (29) can thus be evaluated through the unitary transformation as

$$\psi_n^F = U_F \psi_n \quad (35)$$

$$= U_F U_0^\dagger \psi_n^0 \quad (36)$$

$$= \frac{1}{\sqrt{2^n n!}} \left(\frac{\Omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{\rho(t)}} \left[\frac{u(t) - iv(t)}{\rho(t)}\right]^{n+1/2} \exp\left[\frac{i}{\hbar}(M\dot{x}_p x + \delta(t))\right] \\ \times \exp\left[\frac{(x - x_p)^2}{2\hbar} \left(-\frac{\Omega}{\rho^2(t)} + iM(t)\frac{\dot{\rho}(t)}{\rho(t)}\right)\right] H_n\left(\sqrt{\frac{\Omega}{\hbar}} \frac{x - x_p}{\rho(t)}\right). \quad (37)$$

One can explicitly check that ψ_n^F satisfy the Schrödinger equation

$$O_F \psi_n^F = 0 \quad \text{or} \quad i\hbar \frac{\partial \psi_n^F}{\partial t} = -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x^2} \psi_n^F + \frac{Mw^2}{2} x^2 \psi_n^F - x F(t) \psi_n^F. \quad (38)$$

Through a different approach, relation (35) has long been recognized as in [15, 16] for special cases.

In [10] the wavefunctions for the driven harmonic oscillator are found by factorizing the kernel. If δ is given as

$$\delta = -\frac{M}{2} \frac{\dot{v}}{v} x_p^2 - \frac{1}{2} \int_{t_0}^t M(z) \left(x_p(z) \frac{\dot{v}}{v} - \dot{x}_p(z)\right)^2 dz \quad (39)$$

with an arbitrary constant t_0 , the wavefunctions in equation (37) reduce to those in [10]. And one may easily check that the $\delta(t)$ in equation (39) satisfies the relation (31). The defining relation (31), however, suggests a simpler form $\delta(t)$ as

$$\delta(t) = \int_{t_0}^t \left[\frac{1}{2}M(z)w^2(z)x_p^2(z) - \frac{1}{2}M(z)\dot{x}_p^2(z)\right] dz \quad (40)$$

which can be shown equal to that in equation (39), up to a constant, by making use of the equation of motion in (18).

For a given particular solution $x_p(t)$, new solutions can be obtained by adding linear combinations of homogeneous solutions. For instance, a new solution $x_p'(t)$ can be given as $x_p(t) + Cu(t)$. The $\delta(t)$ depend on the choice of the classical solution, and the difference of δ evaluated with $x_p'(t)$ from that with $x_p(t)$ is written as $-CM\dot{u}(x_p + \frac{1}{2}Cu)$ up to a additive constant.

5. Summary and discussions

In summary, we have found the unitary relations between the systems of time-dependent harmonic oscillators. The first relation is between the systems of time-dependent mass and of unit mass. The second relation is between those of the driven oscillator and the undriven oscillator. Provided the results in equation (13) are given, these relations give a simple method of finding the exact quantum states for a driven harmonic oscillator system [6] or a general quadratic system [10], as shown explicitly with examples. However, a point that should be mentioned is that the unitary relation method cannot give the results in equation (13).

The operator for the first relation is unique up to trivial phase [14], but the other operator which depends on the classical solution is not unique.

Since the operator of the second transformation is an exponential of a linear combination of x and p , the transformation does not change the uncertainties of x and p : to be precise, with the quantum states of $|n; F\rangle$, $|n\rangle$ defined as $\psi_n^F = \langle x|n; F\rangle$, $\psi_n = \langle x|n\rangle$, from equation (35) one can easily prove the relations

$$\langle n; F|(x - \langle n; F|x|n; F\rangle)^2|n; F\rangle = \langle n|(x - \langle n|x|n\rangle)^2|n\rangle \quad (41)$$

$$\langle n; F|(p - \langle n; F|p|n; F\rangle)^2|n; F\rangle = \langle n|(p - \langle n|p|n\rangle)^2|n\rangle. \quad (42)$$

As a final remark, we add a speculation that there might be some relations between a harmonic oscillator system of unit mass time-dependent frequency, and a simple harmonic oscillator. Independently from the time-dependent Hamiltonian system, Gaussian pure states are constructed in [27, 28] in the study of coherent states. The $n = 0$ wavefunctions of all time-dependent harmonic oscillator system belong to those of Gaussian pure states [10]. Our speculation is from the suggestion that the annihilation operator of any Gaussian pure state may be obtained from the operator which annihilates the ground state of a simple harmonic oscillator [28].

References

- [1] Lewis H R 1968 *J. Math. Phys.* **9** 1976
Lewis H R 1967 *Phys. Rev. Lett.* **18** 510
- [2] Lewis H R and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
- [3] Ermakov V 1880 *Univ. Izv. Kiev III* **9** 1
- [4] Ji J-Y, Kim J K, Kim S P and Soh K-S 1995 *Phys. Rev. A* **52** 3352
- [5] Yeon K H, Lee K K, Um C I, George T F and Pandey L N 1993 *Phys. Rev. A* **48** 2716
- [6] Kim H-C, Lee M-H, Ji J Y and Kim J K 1996 *Phys. Rev. A* **53** 3767
- [7] Lee M-H, Kim H-C and Ji J-Y 1977 *J. Korean Phys. Soc.* **31** 3767
- [8] Yeon K H, Kim D H, Um C I, George T F and Pandey L N 1997 *Phys. Rev. A* **55** 4023
- [9] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill) pp 58–60
- [10] Song D-Y 1999 *Phys. Rev. A* **59** 2616
- [11] Leach P G L 1977 *J. Math. Phys.* **18** 1902
Leach P G L 1980 *J. Math. Phys.* **21** 32
- [12] Pedrosa I A 1987 *J. Math. Phys.* **28** 2662
- [13] Li F-I, Wang S J, Weiguny A and Lin D L 1994 *J. Phys. A: Math. Gen.* **27** 985
- [14] Mostafazadeh A 1998 *J. Phys. A: Math. Gen.* **31** 6495
- [15] Husimi K 1953 *Prog. Theor. Phys.* **9** 381
- [16] Haar D T (ed) 1975 *Problems in Quantum Mechanics* (London: Pion) p 169
- [17] Caldirola P 1941 *Nuovo Cimento* **18** 393
- [18] Kanai E 1948 *Prog. Theor. Phys.* **3** 440
- [19] Kerner E H 1958 *Can. J. Phys.* **36** 371
- [20] Maamache M 1996 *J. Phys. A: Math. Gen.* **29** 2833
- [21] Mostafazadeh A 1997 *Phys. Rev. A* **55** 4084
- [22] Hasse R W 1975 *J. Math. Phys.* **16** 2005

- [23] Dodonov V V and Man'ko V I 1979 *Phys. Rev. A* **20** 550
- [24] Colegrave R K and Abdalla M S 1981 *J. Phys. A: Math. Gen.* **14** 2269
Colegrave R K and Abdalla M S 1983 *J. Phys. A: Math. Gen.* **16** 3805
- [25] Pedrosa I A, Serra G P and Guedes I 1997 *Phys. Rev. A* **56** 4300
- [26] Lo C F 1991 *Phys. Rev. A* **43** 404
- [27] Yuen H P 1976 *Phys. Rev. A* **13** 2226
- [28] Schumaker B L 1986 *Phys. Rep.* **135** 317